THEORETICAL STUDY OF HYDRAULIC JUMPS IN STEEP STORM SEWERS

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Abstract: In the design of storm sewers, pipes are often laid on slopes dictated by the actual slope of the terrain, and the slopes can be very large in some urban drainage systems of mountainous areas. The flow in a steep sewer pipe can be easily disturbed, causing an abrupt change from supercritical to subcritical state through a hydraulic jump. Downstream of the hydraulic jump, the flow can come into contact with the pipe vertex, and thus flow choking occurs. This paper focuses on a theoretical analysis of the above-mentioned flow phenomenon involved two connected sewer pipes laid on different slopes. For given values of pipe diameter, discharge, and slope, pertinent variables of flows in the pipes are determined by using the Manning’s formula and the momentum equation. Values of the Froude number, \( Fr \), in response to an increase in the slope of the upstream pipe and/or in discharge from upstream, are calculated. The occurrences of hydraulic jumps as well as their downstream sequent depths are predicted. The maximum filling ratios of upstream flow for which choking occurs at a series of slopes are determined. This paper contributes to practical solutions to the increasingly common problem of waterlogging in urban centres.

1 INTRODUCTION

The Froude number, \( Fr \), is an important parameter in the analysis of free surface flow. The flow is subcritical if \( Fr < 1 \), critical if \( Fr = 1 \), and supercritical if \( Fr > 1 \). The transition from a subcritical to supercritical state is smooth and continuous. Inversely, the change from a supercritical to subcritical state is abrupt and discontinuous through a hydraulic jump (Chow, 1959). In a hilly region, the designs of storm sewer systems are often subject to the constraint of the actual slopes of the terrain. A very steep terrain slope tends to cause supercritical flows in sewer pipes laid on the terrain.

The poor hydraulic performance of supercritical flows in sewer pipes laid on steep slopes is often neglected, which can be easily disturbed by flow perturbations. The perturbations can result from such conditions as changes in flow direction and bottom slope, sewer junctions, or sewer sedimentation (Hager, 1999b). These perturbations may cause hydraulic jumps. If the sequent depth of the jump reaches the sewer soffit, flow choking occurs, and results in a sudden and abrupt transition from free surface flow to pressurized flow.

For given constant discharge and pipe diameter, the steeper the slope, the larger the value of \( Fr \), which will have a higher risk of choking. The upper bound of filling ratio in many design guidelines of storm sewers is approximately 85% (ASCE, 1982). This is based on the concept of uniform flow, without considering the complex flow patterns of supercritical flow. The focus of this paper is on choking phenomenon caused by hydraulic jumps in steep storm sewers.
In the following, calculations are carried out for circular concrete sewers, with an internal diameter of \( d_0 = 0.6 \) m, a Manning’s roughness factor of \( n = 0.013 \). The results are relevant flow variables in response to an increase in the slope of the upstream pipe and/or in discharge from upstream. The results show the condition under which flow choking occurs.

2 METHODS

2.1 Calculations of the Froude Number

For circular pipes, the geometric elements are defined in Figure1:

![Figure1. Geometric elements of a circular section](image)

The angle \( \theta \) is related to the depth of flow, \( h \), and diameter \( d_0 \) as

\[
\theta = \pi - \cos^{-1} \left( \frac{h-\frac{d_0}{2}}{\frac{d_0}{2}} \right) = \pi - \cos^{-1} (2y-1)
\]

where \( y \) is the filling ratio, defined as \( y = h/d_0 \). The flow area, \( A \), is given by

\[
A = \frac{1}{8} (2\theta - \sin 2\theta) d_0^2
\]

The hydraulic depth is related to the angle and pipe diameter as

\[
D = \frac{1}{8} \left( \frac{2\theta - \sin 2\theta}{\sin \theta} \right) d_0
\]

The hydraulic radius varies with the angle and pipe diameter as

\[
R = \frac{1}{4} \left( \frac{1 - \sin 2\theta}{2\theta} \right) d_0
\]

Assume that the sewer pipe is long enough to develop normal flow from upstream. The Manning’s formula can be used to calculate discharge \( Q \):

\[
Q = \frac{1}{n} AR^{2/3} S_0^{1/2}
\]

where \( S_0 \) is the bottom slope.

By definition, the Froude number \( F_r \) is
where \( V \) is the cross-sectionally averaged flow velocity, and \( g \) is the gravitation acceleration (equals to 9.81 m/s\(^2\)).

Expressions for \( Q \) and \( Fr \) in terms of \( \theta \) can be derived by substituting Equations [2], [3], [4] into Equations [5] and [6]:

\[
Q = \frac{1}{n} \left[ \frac{d_0^2}{8} \left( \sin 2\theta \right) \right]^{5/3} (\theta d_0)^{-2/3} S_0^{1/2} \tag{7}
\]

\[
Fr = \frac{g^{-1/2} S_0^{1/2} \left( \sin \theta \right)^{3/2} (2\theta)^{-2/3} \left( 2d_0(2\theta - \sin 2\theta) \right)^{1/6}} \tag{8}
\]

As \( n, g \), and \( d_0 \) are constant variables, the Froude number \( Fr \) only varies with \( \theta \) and \( S_0 \). However, Expression [8] is too complicated for practical applications. A simplified expression for \( Fr \) is introduced

\[
Fr = Q/(gd_0 h^4)^{1/2} \tag{9}
\]

which permits the determination of \( Fr \) from discharge \( Q \), flow depth \( h \), and pipe diameter \( d_0 \) (Hager, 1999b).

Depending on the approach Froude number, hydraulic jumps of different characteristics are possible to occur in sewer pipes. For \( 1 \leq Fr \leq 1.5 \), the jump is an undular jump. For \( 1.5 \leq Fr \leq 2 \), the undulations break at the upstream side. For \( Fr > 2 \), the undulations disappear and direct hydraulic jumps occur (Hager, 1999a).

2.2 Undular hydraulic jump

Undular hydraulic jumps feature successive shockwaves, which are highly sensitive to small perturbations, and hence undesirable. Such jumps may occur in a steep sewer. As illustrated in Figure 2, the main parameters of the jumps are wave crests \( h_{1C}, h_{2C}, \) and \( h_{3C} \), and wave troughs \( h_{1T}, h_{2T}, \) and \( h_{3T} \) along the streamwise direction. Because of a wave diffusion process, the depths of wave peaks decrease in the flow direction (Hager, 1999b). The elevation of the first wave crest is the highest and has the potential to cause flow choking. According to Gargano and Hager (2002), the first wave peak is related to the Froude number \( Fr \) and filling ratio \( y \) of the upstream flow as

\[
h_{1C} = 1.20 Fr_1 y_1^{-0.10} \tag{10}
\]

where the subscript 1 in the Froude number and filling ratio refer to the upstream flow.

Figure 2. Characteristics of undular hydraulic jump in a sloping sewer, showing the elevations of wave crests \( h_{1C}, h_{2C}, \) and \( h_{3C} \), wave troughs \( h_{1T}, h_{2T}, \) and \( h_{3T} \) above the invert of the sewer pipe, and distances \( L_1, L_2, \) and \( L_3 \) between adjacent wave crests. Here, \( h_1 \) is the depth of the upstream flow; \( \alpha \) is the angle between the horizontal and the bottom of the sewer.
2.3 Direct hydraulic jump

In Figure 3, a direct hydraulic jump is illustrated. The flow is supercritical at depth \( h_{j1} \) just before the jump, and is subcritical at depth \( h_{j2} \) just after the jump. The jump has a length of \( L_j \).

![Diagram of a direct hydraulic jump in a sloping sewer.](image)

Figure 3. Direct hydraulic jump in a sloping sewer.

Whether or not the downstream flow is choked depends on \( h_{j2} \), in comparison to the pipe diameter. The condition is similar to the first wave crest of an undular hydraulic jump. Between the upstream cross section (denoted by the subscript 1) and downstream cross section (denoted by the subscript 2), energy loss is unknown a priori, and thus the energy equation cannot be used straightforwardly to solve the hydraulic jump problem. However, the momentum equation can be used to calculate the sequent depth \( h_{j2} \). The momentum equation is:

\[
QV_1 + \rho g Y_{c1} A_1 \cos \alpha = QV_2 + \rho g Y_{c2} A_2 \cos \alpha + \frac{L_j}{A_1 + A_2} \frac{\rho g}{2} \sin \alpha - F_f - F_e
\]

where \( \rho \) is the density of water; \( Y_c \) is the distance from the water surface to the centroid of the cross section; \( \sin \alpha = S_0 \); \( F_f \) is friction force; \( F_e \) is any other external force. Compared to the momentum flux and pressure forces, \( F_f \) can be neglected because \( L_j \) is typically small. It is assumed that no other external forces are involved.

For the purpose of evaluating the gravity term in Equation [11], it is acceptable to assume that \( A_1 = A_2 = \pi d_0^2 / 4 \). With this approximation, Equation [11] can be rewritten as:

\[
\frac{Q_1^2}{gA_1} + Y_{c1} A_1 \sqrt{1-S_0^2} = \frac{Q_2^2}{gA_2} + Y_{c2} A_2 \sqrt{1-S_0^2} + \frac{1}{4} \pi d_0^2 L_j S_0
\]

The term \( Y_c A \) can be determined as:

\[
Y_c A = \frac{d_0^3}{24} (3 \sin \theta - \sin^3 \theta - 3 \theta \cos \theta)
\]

Hager (1999b) suggested an empirical expression of the jump length \( L_j \):

\[
L_j = 1.9 h_{1} \left[ (2e^{1.5 y_1} + e^{-10 S_{y1}-1}) (F_{r1} - 1) \right]
\]

The value of \( \theta_1 \) can be determined for a given discharge \( Q \). Because the parameters \( Fr, h, A \) are only functions of \( \theta, F_{r1}, h_{1}, A_1 \) are known for the upstream flow. Thus, Equation [12] has only one unknown variable \( \theta_2 \), but it cannot be solved explicitly. In this paper, Matlab code was developed and used to find solutions.
3 RESULTS

3.1 The Accuracy of Calculated Froude Number

In Figure 4, the values of $Fr$ based on Equations [7] and [9] are compared. For the usual range of $30\% < y < 80\%$, the relative errors of the approximate $Fr$ values from Equation [9] are less than $\pm 4\%$. This is acceptable.

![Figure 4. Froude number versus filling ratio for $d_0 = 0.6\text{ m}$, $S_0 = 2\%$, and $n = 0.013$.](image)

3.2 The Effect of Bottom Slope on the Froude Number

For a constant value of bottom slope $S_0$, values for the Froude number $Fr$ can be determined, by using [9], for a series values of discharge $Q$. The results of $Fr$ versus $Q$ are shown in Figure 5. For a constant bottom slope $S_0$, the Froude number $Fr$ increases with increasing discharge $Q$ until a certain value $Q$, and then decreases with increasing $Q$. As expected, for the same discharge $Q$, the Froude number $Fr$ increases with increasing slope $S_0$. 

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Note that the Froude number is between 1 and 1.5 for undular jumps, between 1.5 and 2 when the undulations break at the upstream side, and larger than 2 when the undulations disappear and direct hydraulic jumps occur. According to these criteria, flows in storm sewer pipes laid on slopes in the range of $S_0 = 0.005$ to $S_0 = 0.070$ are divided into three categories: 1) For $S_0 < 0.005$, no hydraulic jump occurs; 2) for $0.005 \leq S_0 \leq 0.017$, undular hydraulic jumps are possible to occur; 3) for $0.017 \leq S_0 \leq 0.07$, direct hydraulic jumps are possible.

### 3.3 Downstream filling ratio

The first crest depth $h_{1c}$ for an undular hydraulic jump and the sequent depth $h_{12}$ for a direct hydraulic jump for varying bottom slope can be calculated by using Matlab. And the relation between the downstream filling ratio $y_2$ and upstream filling ratio $y_1$ are graphically shown in Figure 6 and Figure 7. Similar results have been reported earlier in the literatures (Hager, 1999b; Lowe, 2011).
Figure 6. Downstream filling ratio for an undular hydraulic jump versus upstream filling ratio from bottom slope $S_0=0.011$ to $S_0=0.016$.

Figure 7. Downstream filling ratio for a direct hydraulic jump versus upstream filling ratio from bottom slope $S_0=0.020$ to $S_0=0.070$.

3.4 Downstream flow choking

A sewer chokes whenever $h_{1C} \geq d_0$ or $h_{2D} \geq d_0$. Based on Equations [10] and [12], the corresponding filling ratio for choking $y_c$ can be calculated in correspondence to an increase in bottom slope $S_0$. 

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As seen from Figures 8 and 9, the maximum filling ratio for free surface flow to exist downstream of a hydraulic jump in a sewer decreases significantly with increasing bottom slope $S_0$. This means that the steeper the sewer, the higher the risk for choking, as expected. The regions above the curves in Figures 6 and 7 represent the condition that the downstream flows after a hydraulic jump would become pressurized flows.

Such condition is illustrated in Figure 10. The condition is problematic, but has received little attention in the current practice of designing storm sewers. The current practice is that sewers are normally designed for 75%-85% filling ratio, regardless of whether the slopes are mild or steep. In fact, flows in steep sewers may become pressurised caused by hydraulic jumps before reaching 75% filling ratio. In order for free surface flow to be maintained, the upper bound of design filling ratio for steep sewers needs to be reduced appropriately.
4 CONCLUSION

In this paper, hydraulic jumps in steep storm sewers of circular shape were theoretically analysed. The sewer pipes are assumed to have a diameter of $d_0 = 0.6$ m, and a Manning’s roughness factor of $n = 0.013$. In order to clarify flow patterns and hydraulic types for various bottom slope, values of the Froude number in correspondence to increasing slopes from $S_o = 0.005$ to $0.07$ were determined. According to the approach Froude number, there were two main types of hydraulic jumps: For $0.005 \leq S_o \leq 0.017$, possible jumps are mainly an undular hydraulic jump. For $0.017 < S_o \leq 0.07$, possible jumps are a direct hydraulic jump. The depth, $h_{1c}$, below the first wave crest for an undular hydraulic jump and the sequent depth, $h_{j2}$, for a direct hydraulic jump were calculated. Flow choking occurs when $h_{1c} \geq d_0$ or $h_{j2} \geq d_0$. This paper presents useful curves of the maximum filling ratio for free surface flow downstream of a hydraulic jump. These curves delineate the regions where the downstream flows would become pressurized flows, which should be avoided in the design of storm sewers. It have been shown that the common design standard for filling ratio should not be applied to steep sewers directly, because of the risk of flow choking caused by hydraulic jumps.

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References