A THERMODYNAMIC FRAMEWORK FOR CONSTITUTIVE MODELS OF CONCRETE WHICH TAKES INTO ACCOUNT PLASTICITY AND DAMAGE PROPERTIES

D.X.Zhang, W.J. Qu, W.Zhou
Collage of civil engineering, Tongji University, ShangHai, 200092, China

Abstract: Strain softening and stiffness degradation induced by damage and permanent deformation caused by the plasticity were taken into account, and the evolution of damage and plastic variable were used to describe their mechanism. The plasticity and damage mechanics were incorporated into the thermodynamics, four types of energy potential functions and dissipation function, which considered coupling effect of plastic and damage have been developed. Along with flow rule of generalized plasticity, the evolution laws of internal variables were formed, given 32 possible ways of formulating constitutive behavior. The incremental response entirely established by differentiation of the two potentials and by standard matrix manipulation. Finally, the constrained conditions introduced successfully through the Lagrangean penalty functions and strengthened the applications of developed thermodynamic framework.

Key words Thermodynamic; Energy Dissipation; Damage; Constitutive Models; concrete

1 INTRODUCTION

Today, the accurate model of concrete mechanical behaviour under complex loading paths represents a challenging task, especially when the prediction of failure is of interests. Countless theories have been proposed for such materials, based on concepts such as linear and non-linear elasticity, a large number of studies focused on the plasticity and continuum damage mechanics (CDM). The mathematical theory of plasticity is thoroughly established and some of these works were far superior to elastic approaches, but these works failed to address the degradation of the material stiffness due to micro-cracking, (e.g. Bazant ZP,1978; Onate E.,Oliver S. and Lubliner J.,1988; Grassl P., Lundgren K. and Gylltoft K. ,2002; Voyiadjis GZ. and Abu-Lebdeh TM. ,1994). On the other hand, continuum damage mechanics (CDM) has also been used alone with elasticity to construct the material nonlinear behaviour of concrete. However, several facets of concrete behaviour, such as irreversible deformations and inelastic volumetric expansion in compression cannot be analyzed by this approach, (e.g. Krajcinovic D. ,1985; Mazars, J. and Pijaudier-Cabot, G. ,1989; Simo, J.C. and Ju, J.W.,1987a;).Many proposed theories and models for concrete based on plasticity and CDM are inconsistent with the thermodynamics laws. The main motivation of this study is to develop approaches based on the thermodynamic, together with continuum damage mechanics and plasticity theory, in order to achieve a robust framework for constitutive model of concrete.

2 THE PRINCIPLES OF THERMODYNAMICS AND LOCAL STATE METHOD

2.1 Thermodynamic Laws

Following the original work e.g. (Chaboche and Lemaitre, J. 1997), we emphasize the local forms of the First and Second Laws of Thermodynamics for further derivation. Respectively,

[1] $\dot{W}+\dot{Q}=\dot{u}$

[2] $\dot{S}\geq-\left(\frac{q_k}{\theta}\right)_{,k}$

[3] $\theta\dot{s}+\sigma_{ij}\dot{\varepsilon}_{ij}-\dot{u}-\frac{q_k\theta_{,k}}{\theta}\geq 0$

in which $\dot{W}=\sigma_{ij}\dot{\varepsilon}_{ij}$ is the mechanical work input, σ_{ij} is effective stress of point, $\dot{\varepsilon}_{ij}$ is rate of strain; $\dot{Q}=-q_{k,k}$ is the heat supply to a volume element, *s* is the entropy and q_k/θ is denoted as the entropy flux, θ represent the temperature of thermodynamics state , *u* is the internal energy.

2.2 The Local State Method

Concrete under the action of forces, the nonlinear deformation or dissipative characteristics mostly result from the material damage procedure and plastic deformation. In this work, the strain softening and stiffness degradation can be modelled by damage mechanics, while the residual strains and some other macroscopic features are seen to be related to and captured by plasticity theory. In response to the micromechanical processes, the representative macroscopic variables characterizing the material behaviour at microscopic level are the damage indicators $\alpha_{ij}{}^{d}$ (In this study, we restrict ourselves to the case of isotropic damage, so that $\alpha_{ij}{}^{d}$ is simply a scalar internal variable.) and plastic strains($\varepsilon_{ij}{}^{P}$).Thus, for small strain continuum mechanics, the energy functions have been adopted in the following forms: Internal energy: $u=u\left(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},s\right)$; Helmholtz free energy: $f=f\left(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},T\right)$; Enthalpy: $h=h\left(\sigma_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},s\right)$; Gibbs free energy: $g=g\left(\sigma_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},s\right)$. Four general dependent energy forms u, f, h, g can be transformed into each other through the Legrendre transformations. The one of the Legendre transforms among the different energies are given out:

[4] $f\left(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},T\right)=u\left(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},S\right)-s\cdot T$

[5] $h\left(\sigma_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},s\right)=u\left(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},S\right)-\sigma_{ij}\varepsilon_{ij}$

[6] $g\left(\sigma_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},T\right)=f\left(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},T\right)-\sigma_{ij}\varepsilon_{ij}$

Here, Helmholtz free energy, for example, was adopted to derive the basic relationship, and other forms of energy function only for analogy and given corresponding relationship. Differentiation of f and associated with formulation (3), the dissipation function can be achieved:

[7] $d=d^{f}(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},S)=\sigma_{ij}\dot{\varepsilon}_{ij}-\dot{f}-s\dot{\theta}-\frac{q_k\theta_{,k}}{\theta}\geq 0$

Utilizing the assumption that the thermal dissipation $\frac{q_k\theta_{,k}}{\theta}$ is positive and small compared to the mechanical one for slow processes, and hence it can be neglected, the Clausius-Duhem inequality now reduces to a more stringent form :

[8] $d=d^{f}(\varepsilon_{ij},\varepsilon_{ij}{}^{D},\varepsilon_{ij}{}^{P},s)=\sigma_{ij}\dot{\varepsilon}_{ij}-\dot{f}-s\dot{\theta}\geq 0$

with Clausius-Duhem inequality as the following equations:

[9] $\dot{f}=\frac{\partial f}{\partial\varepsilon_{ij}}\dot{\varepsilon}_{ij}+\frac{\partial f}{\partial\theta}\dot{\theta}+\frac{\partial f}{\partial\varepsilon_{ij}{}^{D}}\dot{\varepsilon}_{ij}{}^{D}+\frac{\partial f}{\partial\varepsilon_{ij}{}^{P}}\dot{\varepsilon}_{ij}{}^{P}$

[10] $d=\left(\sigma_{ij}-\frac{\partial f}{\partial\varepsilon_{ij}}\right)\dot{\varepsilon}_{ij}-\left(S+\frac{\partial f}{\partial\theta}\right)\dot{\theta}+\frac{\partial f}{\partial\varepsilon_{ij}{}^{D}}\dot{\varepsilon}_{ij}{}^{D}+\frac{\partial f}{\partial\varepsilon_{ij}{}^{P}}\dot{\varepsilon}_{ij}{}^{P}$

[11] $d=\frac{\partial f}{\partial\varepsilon_{ij}{}^{D}}\dot{\varepsilon}_{ij}{}^{D}+\frac{\partial f}{\partial\varepsilon_{ij}{}^{P}}\dot{\varepsilon}_{ij}{}^{P}\geq 0$

It can also be seen here that the associated variables with strain ε_{ij} and temperature θ are stress θ_{ij} and entropy s respectively. Therefore, in an analogous angle, the thermodynamic forces associated with the internal variables $\varepsilon_{ij}{}^{P}$ and $\varepsilon_{ij}{}^{D}$ can be defined to be

[12] $\overline{\chi}_{ij}^{\,i} = -\dfrac{\partial f}{\partial \varepsilon_{ij}^{\,i}} \quad i = P, D$

[13] $d = \overline{\chi}_{ij}^{\,P}\, \dot{\varepsilon}_{ij}^{\,P} + \overline{\chi}_{ij}^{\,D}\, \dot{\varepsilon}_{ij}^{\,D}$

Houlsby, G.T.and Puzrin, A.M.(2000) regarded to them as generalized stresses. In the previous derivation, the thermodynamic state of the material is considered unrelated to change of internal variables. In fact, things are not a case, for the framework established out of this work have more comprehensive senses and more representatives, the dissipation in the framework is assumed to be a function of the thermodynamics state of the material and the rate of change of the state. Thus, the dissipation function can be expressed in another way here

[14] $d = d^{e}(\varepsilon_{ij}, \varepsilon_{ij}^{\,D}, \varepsilon_{ij}^{\,P}, S; \dot{\varepsilon}_{ij}^{\,D}, \dot{\varepsilon}_{ij}^{\,P}) \geq 0$

In which e stands for any of u, f, h or g by turns (The following section e use the same meaning.), corresponding to the energy expression described by previous section.

In each case we defined “dissipative generalized stress” as $\chi_{ij}^{\,P} = -\partial d^{e} \big/ \partial \dot{\varepsilon}_{ij}^{\,P}$ $\chi_{ij}^{\,D} = -\partial d^{e} \big/ \partial \dot{\varepsilon}_{ij}^{\,D}$, for a rate-independent material the dissipation must be a homogeneous first order function in the rates $\dot{\varepsilon}_{ij}^{\,P}$ and $\dot{\varepsilon}_{ij}^{\,D}$, since the magnitude of dissipated energy must be proportional directly to the magnitude of deformation. For a homogeneous first order function, Euler's theorem gives :

[15] $d = d^{P} + d^{D} = \chi_{j}^{\,P}\, \dot{\varepsilon}_{ij}^{\,P} + \chi_{ij}^{\,D}\, \dot{\varepsilon}_{ij}^{\,D}$

And comparing Eqs (15) we immediately obtain:

[16] $\left(\overline{\chi}_{ij}^{\,i} - \chi_{ij}^{\,i}\right) \dot{\varepsilon}_{ij}^{\,i} = 0 \quad i = P, D$

As $\chi_{ij}^{\,P}$ may be a function of $\dot{\varepsilon}_{ij}^{\,P}$, it can be concluded here that $\left(\overline{\chi}_{ij}^{\,P} - \chi_{ij}^{\,P}\right)$ is always orthogonal to $\dot{\varepsilon}_{ij}^{\,P}$. However, as argued by Ziegler, H. (1983) and Houlsby, G.T.and Puzrin, A.M.(2000), a rather wide range of classification of materials can be described with enforcement of the stronger condition $\overline{\chi}_{ij}^{\,P} = \chi_{ij}^{\,P}$, For the Eq. (16), with the scalar $\dot{\varepsilon}_{ij}^{\,D}$ it is readily seen that $\overline{\chi}_{ij}^{\,D} = \chi_{ij}^{\,D}$ for $\dot{\varepsilon}_{ij}^{\,D} \neq 0$. The connection of the dissipative generalized stress and generalized stress defined formerly, are the key features for obtaining the yield and damage loading functions in this framework.

3 YIELD SURFACE AND FLOW RULE

Since the dissipation function is homogeneous first order, the Legendre transformation is degenerate and the yield function, as the transform of the dissipation function, is the result of this. Then, the yield function could been obtained:

[17] $\chi_{ij}^{\,P}\, \dot{\varepsilon}_{ij}^{\,P} + \chi_{ij}^{\,D}\, \dot{\varepsilon}_{ij}^{\,D} - d^{P} - d^{D} = 0$

In which the dissipation d has been assumed to be decomposed into two parts, as discussed L. Contrafatto and M. Cuomo, (2002), corresponding to the dissipated energies due to plasticity and damage mechanisms. For rate-independent materials, the Legendre transformation of d^{P} and presents us the yield and damage functions

[18] $\lambda^{i} F^{i} = \chi_{ij}^{\,i}\, \dot{\varepsilon}_{ij}^{\,i} - d^{i} = \left(\chi_{ij}^{\,i} - \dfrac{\partial d^{i}}{\partial \dot{\varepsilon}_{ij}^{\,i}}\right) \dot{\varepsilon}_{ij}^{\,i} = 0 \qquad (i = P, D)$

where λ^{i} $(i = P, D)$ is an arbitrary non-negative multiplier and, depending on the dissipation of energy function, F^{P}, F^{P} can be defined as follows in four forms, respectively.

[19] $F^{i} = F^{ei}\left(\varepsilon_{ij}, \varepsilon_{ij}^{\,i}, S, \chi_{ij}^{\,i}\right) = 0 \qquad (i = P, D)$

The differential of the yield function gives the flow rule:

[20] $\dot{\varepsilon}_{ij}^{\,i} = \lambda^i \dfrac{\partial F^{ei}}{\partial \chi_{ij}^{\,i}} \qquad (i = P, D)$

As discussed below, $\dot{\varepsilon}_{ij}^{\,P}$ and $\dot{\varepsilon}_{ij}^{\,D}$ can be identified with the conventionally defined plastic strain rates and damage strain rates, respectively. Since $d = d^P + d^D = \chi_j^{\,P}\, \dot{\varepsilon}_{ij}^{\,P} + \chi_{ij}^{\,D}\, \dot{\varepsilon}_{ij}^{\,D} \geq 0$ it follows that the condition on F^{eP} and F^{eD} is $\chi_{ij}^{\,P}\, \partial F^{eP} \big/ \partial \chi_{ij}^{\,P} \geq 0$ and $\chi_{ij}^{\,D}\, \partial F^{eD} \big/ \partial \chi_{ij}^{\,D} \geq 0$ (since $\lambda^P \geq 0, \lambda^D \geq 0$). This has a straightforward geometric interpretation, on the condition that the function F^{eP} and F^{eD} contains the origin in generalized stress space and satisfies certain convexity conditions. It does not require, however, that the yield surface should be strictly convex either in generalized stress space or in stress space.

For each of the functions e (u, f, h or g) a further transformation is possible, changing the independent variable from $\varepsilon_{ij}^P, \varepsilon_{ij}^D$ to $\bar{\chi}_{ij}^P, \bar{\chi}_{ij}^D$, respectively, in the formulation $\bar{e} = e + \bar{\chi}_{ij}^P \varepsilon_{ij}^P + \bar{\chi}_{ij}^D \varepsilon_{ij}^D$. Correspondingly the relevant passive variable in d^{eP}, d^{eD} or F^{eP}, F^{eD} is changed from ($\varepsilon_{ij}^P, \varepsilon_{ij}^D$ to $\bar{\chi}_{ij}^P \;\; \bar{\chi}_{ij}^D$). After the transformation notes the results $\varepsilon_{ij}^P = \partial \bar{e} \big/ \partial \bar{\chi}_{ij}^P \;\; \varepsilon_{ij}^D = \partial \bar{e} \big/ \partial \bar{\chi}_{ij}^D$ and $\partial e / \partial x_{(ij)} = \partial \bar{e} / \partial x_{(ij)}$, where $x_{(ij)}$ is any of the passive variables ε_{ij}, σ_{ij}, $\varepsilon_{ij}^{\,P}$, $\varepsilon_{ij}^{\,D}$, s or θ. This last result gives alternative forms for the differentiation to obtain the appropriate complementary variables.

4 A COMPLETE CONSTITUTIVE FORMULATIONS

Table 1 : All 32 possible formulations of dissipation and yield function

Energy function		u or $\bar{u}$	f or $\bar{f}$	h or $\bar{h}$	g or $\bar{g}$
Dissi-pation function $d^e \geq 0$	ε_{ij}^P	$u(\varepsilon_{ij}, \varepsilon_{ij}^P, s)$ $d^u(\varepsilon_{ij}, \varepsilon_{ij}^P, s, \dot{\varepsilon}_{ij}^P)$	$f(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta)$ $d^f(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta, \dot{\varepsilon}_{ij}^P)$	$h(\varepsilon_{ij}, \varepsilon_{ij}^P, s)$ $d^h(\varepsilon_{ij}, \varepsilon_{ij}^P, s, \dot{\varepsilon}_{ij}^P)$	$g(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta)$ $d^g(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta, \dot{\varepsilon}_{ij}^P)$
	ε_{ij}^D	$u(\varepsilon_{ij}, \varepsilon_{ij}^D, s)$ $d^u(\varepsilon_{ij}, \varepsilon_{ij}^D, s, \dot{\varepsilon}_{ij}^D)$	$f(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta)$ $d^f(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta, \dot{\varepsilon}_{ij}^D)$	$h(\varepsilon_{ij}, \varepsilon_{ij}^D, s)$ $d^h(\varepsilon_{ij}, \varepsilon_{ij}^D, s, \dot{\varepsilon}_{ij}^D)$	$g(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta)$ $d^g(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta, \dot{\varepsilon}_{ij}^D)$
	$\bar{\chi}_{ij}^P$	$\bar{u}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s)$ $d^{\bar{u}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s, \dot{\varepsilon}_{ij}^P)$	$\bar{f}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta)$ $d^{\bar{f}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta, \dot{\varepsilon}_{ij}^P)$	$\bar{h}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s)$ $d^{\bar{h}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s, \dot{\varepsilon}_{ij}^P)$	$\bar{g}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta)$ $d^{\bar{g}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta, \dot{\varepsilon}_{ij}^P)$
	$\bar{\chi}_{ij}^D$	$\bar{u}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s)$ $d^{\bar{u}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s, \dot{\varepsilon}_{ij}^D)$	$\bar{f}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta)$ $d^{\bar{f}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta, \dot{\varepsilon}_{ij}^D)$	$\bar{h}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s)$ $d^{\bar{h}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s, \dot{\varepsilon}_{ij}^P)$	$\bar{g}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta)$ $d^{\bar{g}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta, \dot{\varepsilon}_{ij}^D)$
Yield surface $F^e \geq 0$	ε_{ij}^P	$u(\varepsilon_{ij}, \varepsilon_{ij}^P, s)$ $F^u(\varepsilon_{ij}, \varepsilon_{ij}^P, s, \chi_{ij}^P)$	$f(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta)$ $F^f(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta, \chi_{ij}^P)$	$h(\varepsilon_{ij}, \varepsilon_{ij}^P, s)$ $F^h(\varepsilon_{ij}, \varepsilon_{ij}^P, s, \chi_{ij}^P)$	$g(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta)$ $F^g(\varepsilon_{ij}, \varepsilon_{ij}^P, \theta, \chi_{ij}^P)$
	ε_{ij}^D	$u(\varepsilon_{ij}, \varepsilon_{ij}^D, s)$ $F^u(\varepsilon_{ij}, \varepsilon_{ij}^D, s, \chi_{ij}^D)$	$f(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta)$ $F^f(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta, \chi_{ij}^D)$	$h(\varepsilon_{ij}, \varepsilon_{ij}^D, s)$ $F^h(\varepsilon_{ij}, \varepsilon_{ij}^D, s, \chi_{ij}^D)$	$g(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta)$ $F^g(\varepsilon_{ij}, \varepsilon_{ij}^D, \theta, \chi_{ij}^D)$
	$\bar{\chi}_{ij}^P$	$\bar{u}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s)$ $F^{\bar{u}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s, \chi_{ij}^P)$	$\bar{f}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta)$ $F^{\bar{f}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta, \chi_{ij}^P)$	$\bar{h}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s)$ $F^{\bar{h}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, s, \chi_{ij}^P)$	$\bar{g}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta)$ $F^{\bar{g}}(\varepsilon_{ij}, \bar{\chi}_{ij}^P, \theta, \chi_{ij}^P)$
	$\bar{\chi}_{ij}^D$	$\bar{u}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s)$ $F^{\bar{u}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s, \chi_{ij}^D)$	$\bar{f}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta)$ $F^{\bar{f}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta, \chi_{ij}^D)$	$\bar{h}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s)$ $F^{\bar{h}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, s, \chi_{ij}^D)$	$\bar{g}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta)$ $F^{\bar{g}}(\varepsilon_{ij}, \bar{\chi}_{ij}^D, \theta, \chi_{ij}^D)$

Adopting the approach described above, the constitutive behaviour is entirely defined as the specification of two potentials: energy potential and dissipation function or the yield surface. There are a total of 32 different possibilities, however, for the choice of the potentials, representing all permutations on the following possibilities:choice of u, f, h or g for the energy function, dissipation function d^{eP}, d^{eD} or yield surface F^{eP}, F^{eD}, and transformation between ε_{ij}^{P}, ε_{ij}^{D} and $\overline{\chi}_{ij}^{P}$, $\overline{\chi}_{ij}^{D}$ for the energy function.

The possibilities are illustrated in Table 1. In principle any of the 32 formulations could be used to provide a complete specification of the constitutive behaviour of the material.

Table 2: Constitutive equations on differentiation of energy and dissipation/yield function

Energy function		u or $\bar{u}$	f or $\bar{f}$	h or $\bar{h}$	g or $\bar{g}$
Dissip-ation function $d^e \ge 0$		$\sigma_{ij} = \partial u / \partial \varepsilon_{ij}$	$\sigma_{ij} = \partial f / \partial \varepsilon_{ij}$	$\varepsilon_{ij} = -\partial u / \partial \sigma_{ij}$	$\varepsilon_{ij} = -\partial g / \partial \sigma_{ij}$
		$\theta = \partial u / \partial s$	$s = -\partial f / \partial \theta$	$\theta = \partial h / \partial s$	$s = -\partial g / \partial \theta$
	ε_{ij}^{P}	$\overline{\chi}_{ij}^{P} = -\partial u / \partial \varepsilon_{ij}^{P}$	$\overline{\chi}_{ij}^{P} = -\partial f / \partial \varepsilon_{ij}^{P}$	$\overline{\chi}_{ij}^{P} = -\partial h / \partial \varepsilon_{ij}^{P}$	$\overline{\chi}_{ij}^{P} = -\partial g / \partial \varepsilon_{ij}^{P}$
		$\chi_{ij}^{P} = \partial d^{u} / \partial \dot{\varepsilon}_{ij}^{P}$	$\chi_{ij}^{P} = \partial d^{f} / \partial \dot{\varepsilon}_{ij}^{P}$	$\chi_{ij}^{P} = \partial d^{h} / \partial \dot{\varepsilon}_{ij}^{P}$	$\chi_{ij}^{P} = \partial d^{g} / \partial \dot{\varepsilon}_{ij}^{P}$
	ε_{ij}^{D}	$\overline{\chi}_{ij}^{D} = -\partial u / \partial \varepsilon_{ij}^{D}$	$\overline{\chi}_{ij}^{D} = -\partial f / \partial \varepsilon_{ij}^{D}$	$\overline{\chi}_{ij}^{D} = -\partial h / \partial \varepsilon_{ij}^{D}$	$\overline{\chi}_{ij}^{D} = -\partial g / \partial \varepsilon_{ij}^{D}$
		$\chi_{ij}^{D} = \partial d^{u} / \partial \dot{\varepsilon}_{ij}^{D}$	$\chi_{ij}^{D} = \partial d^{f} / \partial \dot{\varepsilon}_{ij}^{D}$	$\chi_{ij}^{D} = \partial d^{h} / \partial \dot{\varepsilon}_{ij}^{D}$	$\chi_{ij}^{D} = \partial d^{g} / \partial \dot{\varepsilon}_{ij}^{D}$
		$\sigma_{ij} = \partial \bar{u} / \partial \varepsilon_{ij}$	$\sigma_{ij} = \partial \bar{f} / \partial \varepsilon_{ij}$	$\varepsilon_{ij} = -\partial \bar{h} / \partial \sigma_{ij}$	$\varepsilon_{ij} = -\partial \bar{g} / \partial \sigma_{ij}$
		$\theta = \partial \bar{u} / \partial s$	$s = -\partial \bar{f} / \partial \theta$	$\theta = \partial \bar{h} / \partial s$	$s = -\partial \bar{g} / \partial \theta$
	$\overline{\chi}_{ij}^{P}$	$\varepsilon_{ij}^{P} = \partial \bar{u} / \partial \overline{\chi}_{ij}^{P}$	$\varepsilon_{ij}^{P} = \partial \bar{f} / \partial \overline{\chi}_{ij}^{P}$	$\varepsilon_{ij}^{P} = \partial \bar{h} / \partial \overline{\chi}_{ij}^{P}$	$\varepsilon_{ij}^{P} = \partial \bar{g} / \partial \overline{\chi}_{ij}^{P}$
		$\chi_{ij}^{P} = \partial d^{\bar{u}} / \partial \dot{\varepsilon}_{ij}^{P}$	$\chi_{ij}^{P} = \partial d^{\bar{f}} / \partial \dot{\varepsilon}_{ij}^{P}$	$\chi_{ij}^{P} = \partial d^{\bar{h}} / \partial \dot{\varepsilon}_{ij}^{P}$	$\chi_{ij}^{P} = \partial d^{\bar{g}} / \partial \dot{\alpha}_{ij}^{P}$
	$\overline{\chi}_{ij}^{D}$	$\varepsilon_{ij}^{D} = \partial \bar{u} / \partial \overline{\chi}_{ij}^{D}$	$\varepsilon_{ij}^{D} = \partial \bar{f} / \partial \overline{\chi}_{ij}^{D}$	$\varepsilon_{ij}^{D} = \partial \bar{h} / \partial \overline{\chi}_{ij}^{D}$	$\varepsilon_{ij}^{D} = \partial \bar{g} / \partial \overline{\chi}_{ij}^{D}$
		$\chi_{ij}^{D} = \partial d^{\bar{u}} / \partial \dot{\varepsilon}_{ij}^{D}$	$\chi_{ij}^{D} = \partial d^{\bar{f}} / \partial \dot{\varepsilon}_{ij}^{D}$	$\chi_{ij}^{D} = \partial d^{\bar{h}} / \partial \dot{\varepsilon}_{ij}^{D}$	$\chi_{ij}^{D} = \partial d^{\bar{g}} / \partial \dot{\alpha}_{ij}^{D}$
Yield surface $F^e \ge 0$		$\sigma_{ij} = \partial u / \partial \varepsilon_{ij}$	$\sigma_{ij} = \partial f / \partial \varepsilon_{ij}$	$\varepsilon_{ij} = -\partial h / \partial \sigma_{ij}$	$\varepsilon_{ij} = -\partial g / \partial \sigma_{ij}$
		$\theta = \partial u / \partial s$	$s = -\partial f / \partial \theta$	$\theta = \partial h / \partial s$	$s = -\partial g / \partial \theta$
	ε_{ij}^{P}	$\overline{\chi}_{ij}^{P} = -\partial u / \partial \varepsilon_{ij}^{P}$	$\overline{\chi}_{ij}^{P} = -\partial f / \partial \varepsilon_{ij}^{P}$	$\overline{\chi}_{ij}^{P} = -\partial h / \partial \varepsilon_{ij}^{P}$	$\overline{\chi}_{ij}^{P} = -\partial g / \partial \varepsilon_{ij}^{P}$
		$\dot{\varepsilon}_{ij}^{P} = \lambda^{P} \cdot \partial y^{u} / \partial \chi_{ij}^{P}$	$\dot{\varepsilon}_{ij}^{P} = \lambda^{P} \cdot \partial y^{f} / \partial \chi_{ij}^{P}$	$\dot{\varepsilon}_{ij}^{P} = \lambda^{P} \cdot \partial y^{h} / \partial \chi_{ij}^{P}$	$\dot{\varepsilon}_{ij}^{P} = \lambda^{P} \cdot \partial y^{g} / \partial \chi_{ij}^{P}$
	ε_{ij}^{D}	$\overline{\chi}_{ij}^{D} = -\partial u / \partial \varepsilon_{ij}^{D}$	$\overline{\chi}_{ij}^{D} = -\partial f / \partial \varepsilon_{ij}^{D}$	$\overline{\chi}_{ij}^{D} = -\partial h / \partial \varepsilon_{ij}^{D}$	$\overline{\chi}_{ij}^{P} = -\partial g / \partial \varepsilon_{ij}^{P}$
		$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{u} / \partial \chi_{ij}^{D}$	$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{f} / \partial \chi_{ij}^{D}$	$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{h} / \partial \chi_{ij}^{D}$	$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{g} / \partial \chi_{ij}^{D}$
		$\sigma_{ij} = \partial \bar{u} / \partial \varepsilon_{ij}$	$\sigma_{ij} = \partial \bar{f} / \partial \varepsilon_{ij}$	$\varepsilon_{ij} = -\partial \bar{h} / \partial \sigma_{ij}$	$\varepsilon_{ij} = -\partial \bar{g} / \partial \sigma_{ij}$
		$\theta = \partial \bar{u} / \partial s$	$s = -\partial \bar{f} / \partial \theta$	$\theta = \partial \bar{h} / \partial s$	$s = -\partial \bar{h} / \partial \theta$
	ε_{ij}^{P}	$\varepsilon_{ij}^{P} = \partial \bar{u} / \partial \overline{\chi}_{ij}^{P}$	$\varepsilon_{ij}^{P} = \partial \bar{f} / \partial \overline{\chi}_{ij}^{P}$	$\varepsilon_{ij}^{P} = \partial \bar{h} / \partial \overline{\chi}_{ij}^{P}$	$\varepsilon_{ij}^{P} = \partial \bar{g} / \partial \overline{\chi}_{ij}^{P}$
		$\dot{\varepsilon}_{ij}^{P} = \lambda \cdot \partial y^{\bar{u}} / \partial \chi_{ij}^{P}$	$\dot{\varepsilon}_{ij}^{P} = \lambda^{P} \cdot \partial y^{\bar{f}} / \partial \chi_{ij}^{P}$	$\dot{\varepsilon}_{ij}^{P} = \lambda^{P} \cdot \partial y^{\bar{h}} / \partial \chi_{ij}^{P}$	$\dot{\varepsilon}_{ij}^{P} = \lambda^{P} \cdot \partial y^{\bar{g}} / \partial \chi_{Ij}^{P}$
	ε_{ij}^{D}	$\varepsilon_{ij}^{D} = \partial \bar{u} / \partial \overline{\chi}_{ij}^{D}$	$\varepsilon_{ij}^{D} = \partial \bar{f} / \partial \overline{\chi}_{ij}^{D}$	$\varepsilon_{ij}^{D} = \partial \bar{h} / \partial \overline{\chi}_{ij}^{D}$	$\varepsilon_{ij}^{D} = \partial \bar{g} / \partial \overline{\chi}_{ij}^{D}$
		$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{\bar{u}} / \partial \chi_{ij}^{D}$	$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{\bar{f}} / \partial \chi_{ij}^{D}$	$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{\bar{h}} / \partial \chi_{ij}^{D}$	$\dot{\varepsilon}_{ij}^{D} = \lambda^{D} \cdot \partial y^{\bar{g}} / \partial \chi_{Ij}^{D}$

On differentiating the energy function and dissipation or yield functions with respect to the appropriate variables, the relationships in Table 2 are obtained. Once the chosen two scalar functions have been specified, the entire constitutive behaviour can be derived from the differentials in the appropriate box in Table 2, together with the condition $\chi_{ij} = \overline{\chi}_{ij}$.

5 INCREMENTAL RESPONSES

The result is generalized to show how incremental response could be applied to any model in the (f, d^f) formulation, subject to a restriction on the form of the dissipation function. Differentiation of the energy expressions in Table 2 leads directly to the results in Table 2 where the (symmetric) matrix $[u'']$ is defined as follows:

Table 3 Incremental constitutive equations

u or $\overline{u}$	f or $\overline{f}$	h or $\overline{h}$	g or $\overline{g}$
$\begin{Bmatrix} \dot{\sigma}_{ij} \\ \dot{\theta} \\ -\dot{\chi}^P_{ij} \\ -\dot{\chi}^D_{ij} \end{Bmatrix} = [u''] \begin{Bmatrix} \dot{\varepsilon}_{kl} \\ \dot{s} \\ \dot{\varepsilon}^P_{kl} \\ \dot{\varepsilon}^D_{kl} \end{Bmatrix}$	$\begin{Bmatrix} \dot{\sigma}_{ij} \\ -\dot{s} \\ -\dot{\overline{\chi}}_{ij} \\ -\dot{\overline{\chi}}_{ij} \end{Bmatrix} = [f''] \begin{Bmatrix} \dot{\varepsilon}_{kl} \\ \dot{\theta} \\ \dot{\varepsilon}^P_{kl} \\ \dot{\varepsilon}^D_{kl} \end{Bmatrix}$	$\begin{Bmatrix} -\dot{\varepsilon}_{ij} \\ \dot{\theta} \\ -\dot{\overline{\chi}}^P_{ij} \\ -\dot{\overline{\chi}}^D_{ij} \end{Bmatrix} = [h''] \begin{Bmatrix} \dot{\sigma}_{kl} \\ \dot{s} \\ \dot{\varepsilon}^P_{kl} \\ \dot{\alpha}^P_{kl} \end{Bmatrix}$	$\begin{Bmatrix} -\dot{\varepsilon}_{ij} \\ -\dot{s} \\ -\dot{\overline{\chi}}^P_{ij} \\ -\dot{\overline{\chi}}^D_{ij} \end{Bmatrix} = [g''] \begin{Bmatrix} \dot{\sigma}_{kl} \\ \dot{\theta} \\ \dot{\varepsilon}^P_{kl} \\ \dot{\alpha}^D_{kl} \end{Bmatrix}$
$\begin{Bmatrix} \dot{\sigma}_{ij} \\ \dot{\theta} \\ \dot{\varepsilon}^P_{ij} \\ \dot{\alpha}^D_{ij} \end{Bmatrix} = \left[\overline{u}''\right] \begin{Bmatrix} \dot{\varepsilon}_{kl} \\ \dot{s} \\ \dot{\overline{\chi}}^P_{kl} \\ \dot{\overline{\chi}}^D_{kl} \end{Bmatrix}$	$\begin{Bmatrix} \dot{\sigma}_{ij} \\ -\dot{s} \\ \dot{\varepsilon}^P_{ij} \\ \dot{\alpha}^D_{ij} \end{Bmatrix} = \left[\overline{f}''\right] \begin{Bmatrix} \dot{\varepsilon}_{kl} \\ \dot{\theta} \\ \dot{\overline{\chi}}^P_{kl} \\ \dot{\overline{\chi}}^D_{kl} \end{Bmatrix}$	$\begin{Bmatrix} -\dot{\varepsilon}_{ij} \\ \dot{\theta} \\ \dot{\varepsilon}^P_{ij} \\ \dot{\varepsilon}^D_{ij} \end{Bmatrix} = \left[\overline{h}''\right] \begin{Bmatrix} \dot{\sigma}_{kl} \\ \dot{s} \\ \dot{\overline{\chi}}^P_{kl} \\ \dot{\overline{\chi}}^D_{kl} \end{Bmatrix}$	$\begin{Bmatrix} -\dot{\varepsilon}_{ij} \\ -\dot{s} \\ \dot{\varepsilon}^P_{ij} \\ \dot{\varepsilon}^D_{ij} \end{Bmatrix} = \left[\overline{g}''\right] \begin{Bmatrix} \dot{\sigma}_{kl} \\ \dot{\theta} \\ \dot{\overline{\chi}}^P_{kl} \\ \dot{\overline{\chi}}^D_{kl} \end{Bmatrix}$

$$[21] \quad [u''] = \begin{bmatrix} \frac{\partial^2 u}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} & \frac{\partial^2 u}{\partial \varepsilon_{ij} \partial s} & \frac{\partial^2 u}{\partial \varepsilon_{ij} \partial \varepsilon^P_{kl}} & \frac{\partial^2 u}{\partial \varepsilon_{ij} \partial \varepsilon^D_{kl}} \\ \frac{\partial^2 u}{\partial s \partial \varepsilon_{kl}} & \frac{\partial^2 u}{\partial s \partial s} & \frac{\partial^2 u}{\partial s \partial \varepsilon^P_{kl}} & \frac{\partial^2 u}{\partial s \partial \varepsilon^D_{kl}} \\ \frac{\partial^2 u}{\partial \varepsilon^P_{ij} \partial \varepsilon_{kl}} & \frac{\partial^2 u}{\partial \varepsilon^P_{ij} \partial s} & \frac{\partial^2 u}{\partial \varepsilon^P_{ij} \partial \varepsilon^P_{kl}} & \frac{\partial^2 u}{\partial \varepsilon^P_{ij} \partial \varepsilon^D_{kl}} \\ \frac{\partial^2 u}{\partial \varepsilon^D_{ij} \partial \varepsilon_{kl}} & \frac{\partial^2 u}{\partial \varepsilon^D_{ij} \partial s} & \frac{\partial^2 u}{\partial \varepsilon^D_{ij} \partial \varepsilon^P_{kl}} & \frac{\partial^2 u}{\partial \varepsilon^D_{ij} \partial \varepsilon^D_{kl}} \end{bmatrix}$$

$$[22] \quad \left[\overline{u}''\right] = \begin{bmatrix} \frac{\partial^2 \overline{u}}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} & \frac{\partial^2 \overline{u}}{\partial \varepsilon_{ij} \partial s} & \frac{\partial^2 \overline{u}}{\partial \varepsilon_{ij} \partial \overline{\chi}^P_{kl}} & \frac{\partial^2 \overline{u}}{\partial \varepsilon_{ij} \partial \overline{\chi}^D_{kl}} \\ \frac{\partial^2 \overline{u}}{\partial s \partial \varepsilon_{kl}} & \frac{\partial^2 \overline{u}}{\partial s \partial s} & \frac{\partial^2 \overline{u}}{\partial s \partial \overline{\chi}^P_{kl}} & \frac{\partial^2 \overline{u}}{\partial s \partial \overline{\chi}^D_{kl}} \\ \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^P_{ij} \partial \varepsilon_{kl}} & \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^P_{ij} \partial s} & \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^P_{ij} \partial \overline{\chi}^P_{kl}} & \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^P_{ij} \partial \overline{\chi}^D_{kl}} \\ \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^D_{ij} \partial \varepsilon_{kl}} & \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^D_{ij} \partial s} & \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^D_{ij} \partial \overline{\chi}^P_{kl}} & \frac{\partial^2 \overline{u}}{\partial \overline{\chi}^D_{ij} \partial \overline{\chi}^D_{kl}} \end{bmatrix}$$

and the matrices $[f'']$, $\left[\overline{f}''\right]$, $[h'']$, $\left[\overline{h}''\right]$, $[g'']$ and $\left[\overline{g}''\right]$ are similarly defined, with appropriate permutation of the energy functions and independent variables.
These incremental relationships are true of both dissipation and yield function formulations. However, in general the explicit stress-strain response can be only obtained in those formulations based on the yield

functions and only of the F^e type, i.e. only for the formulations shown in the secondly last row of Tables 2 and 3. For each of these forms the incremental relationships can be interpreted (noting that $\dot{\bar{\chi}}_{ij} = \dot{\chi}_{ij}$) in the following form:

$$[23] \begin{Bmatrix} \dot{a}_{ij} \\ \dot{x} \\ -\dot{\chi}^P_{ij} \\ -\dot{\chi}^D_{ij} \end{Bmatrix} = \begin{bmatrix} \frac{\partial^2 e}{\partial b_{ij}\partial b_{kl}} & \frac{\partial^2 e}{\partial b_{ij}\partial z} & \frac{\partial^2 e}{\partial b_{ij}\partial \varepsilon^P_{kl}} & \frac{\partial^2 e}{\partial b_{ij}\partial \varepsilon^D_{kl}} \\ \frac{\partial^2 e}{\partial z\partial b_{kl}} & \frac{\partial^2 e}{\partial z\partial z} & \frac{\partial^2 e}{\partial z\partial \varepsilon^P_{kl}} & \frac{\partial^2 e}{\partial z\partial \varepsilon^D_{kl}} \\ \frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial b_{kl}} & \frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial z} & \frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial \varepsilon^P_{kl}} & \frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial \varepsilon^D_{kl}} \\ \frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial b_{kl}} & \frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial z} & \frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial \varepsilon^P_{kl}} & \frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial \varepsilon^D_{kl}} \end{bmatrix} \begin{Bmatrix} \dot{b}_{kl} \\ \dot{z} \\ -\dot{\varepsilon}^P_{kl} \\ -\dot{\varepsilon}^P_{kl} \end{Bmatrix}$$

Talbe 4 substitution of variables for different formulation

e	u	f	h	g
a_{ij}	σ_{ij}	σ_{ij}	$-\varepsilon_{ij}$	$-\varepsilon_{ij}$
b_{ij}	ε_{ij}	ε_{ij}	σ_{ij}	σ_{ij}
x	θ	$-s$	θ	$-s$
z	s	θ	s	θ

Where substitutions for e, α_{ij}, b_{ij}, x and z are to be taken as the appropriate column of Table 4. Eq. (21) is used together with the flow rule:

$$[24]\ \dot{\varepsilon}^i_{ij} = \lambda^{ie}\frac{\partial F^{ie}}{\partial \chi^{ie}_{ij}} \qquad (i = P, D)$$

The infinitesimal multiplier λ^P and λ^D are obtained by substituting of the above equations into the consistency condition, which is obtained by differentiation of the yield function:

$$[25]\ \dot{F}^{ie} = \frac{\partial F^{ie}}{\partial b_{ij}}\dot{b}_{ij} + \frac{\partial F^{ie}}{\partial \varepsilon^i_{ij}}\dot{\varepsilon}^i_{ij} + \frac{\partial F^{ie}}{\partial z}\dot{z} + \frac{\partial F^{ie}}{\partial \chi^i_{ij}}\dot{\chi}^i_{ij} = 0 \qquad (i = P, D)$$

Together with the orthogonal condition in its incremental form:

$$[26]\ \dot{\chi}^D_{ij} = \dot{\bar{\chi}}^D_{ij} = -\frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl} - \frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial s}\dot{s} - \frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial \varepsilon^P_{ij}}\dot{\varepsilon}^P_{ij} - \frac{\partial^2 e}{\partial \varepsilon^D_{ij}\partial \varepsilon^D_{ij}}\dot{\varepsilon}^D_{ij}$$

$$[27]\ \dot{\chi}^P_{ij} = \dot{\bar{\chi}}^P_{ij} = -\frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial \varepsilon_{kl}}\dot{\varepsilon}_{kl} - \frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial s}\dot{s} - \frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial \varepsilon^P_{ij}}\dot{\varepsilon}^P_{ij} - \frac{\partial^2 e}{\partial \varepsilon^P_{ij}\partial \varepsilon^D_{ij}}\dot{\varepsilon}^D_{ij}$$

and $\frac{\partial^2 u}{\partial \varepsilon^D_{ij}\partial \varepsilon^P_{ij}}\dot{\varepsilon}^P_{ij} = 0$ (To concise the derivation of the formula, this article assumes that the plasticity of material does not affect the damage properties, but damage of materials can contribute to the plasticity), those can be used to calculate:

$$[28]\ \lambda^{ie} = -\frac{A^{i,\ eb}_{ij}}{B^{i,e}}\dot{b}_{ij} - \frac{A^{i,ez}}{B^{i,e}}\dot{z} \qquad (i = P, D)$$

where for convenience we define the notation:

[29] $A^{D,eb}_{ij} = \frac{\partial F^{De}}{\partial b_{ij}} - \frac{\partial F^{De}}{\partial \chi^D_{ij}} \frac{\partial^2 e}{\partial \varepsilon^D_{ij} \partial b_{ij}}$.

[30] $A^{D,ez} = \frac{\partial F^{De}}{\partial z} - \frac{\partial F^{De}}{\partial \chi^D_{ij}} \frac{\partial^2 e}{\partial \varepsilon^D_{ij} \partial z}$.

[31] $B^{D,e} = \left(\frac{\partial F^{De}}{\partial \varepsilon^D_{ij}} - \frac{\partial F^{De}}{\partial \chi^D_{kl}} \frac{\partial^2 e}{\partial \varepsilon^D_{kl} \partial \varepsilon^D_{ij}} \right) \frac{\partial F^{De}}{\partial \chi^D_{ij}}$.

[32] $A^{P,eb}_{ij} = \frac{\partial F^P}{\partial b_{ij}} - \frac{\partial F^P}{\partial \chi^P_{ij}} \frac{\partial^2 e}{\partial \varepsilon^P_{ij} \partial b_{ij}} - \frac{A^{D,eb}_{ij}}{B^{D,e}} \left(\frac{\partial F^P}{\partial \varepsilon^D_{ij}} \frac{\partial F^p}{\partial \chi^D_{ij}} \right)$.

[33] $A^{P,ez} = \frac{\partial F^P}{\partial z} - \frac{\partial F^P}{\partial \chi^P_{ij}} \frac{\partial^2 e}{\partial \varepsilon^P_{ij} \partial z} - \frac{A^{D,ez}}{B^{D,e}} \left(\frac{\partial F^P}{\partial \varepsilon^D_{ij}} \frac{\partial F^D}{\partial \chi^D_{ij}} \right)$.

[34] $B^{P,e} = \frac{\partial F^P}{\partial \chi^P_{ij}} \frac{\partial^2 e}{\partial \varepsilon^P_{ij} \partial \varepsilon^P_{ij}} \frac{\partial F^P}{\partial \chi^P_{ij}}$.

This leads to the following incremental stress-strain relationships:

[35] $$\begin{Bmatrix} \dot{a}_{ij} \\ \dot{x} \\ -\dot{\chi}^D_{ij} \\ \dot{\varepsilon}^D_{ij} \\ -\dot{\chi}^P_{ij} \\ \dot{\varepsilon}^P_{ij} \\ \lambda^D \\ \lambda^D \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 e}{\partial b_{ij} \partial b_{kl}} - \frac{\partial^2 e}{\partial b_{ij} \partial \varepsilon^D_{kl}} {}^{D1}C^{eb}_{mnkl} + \frac{\partial^2 e}{\partial b_{ij} \partial \varepsilon^P_{kl}} {}^{P1}C^{eb}_{mnkl} & \frac{\partial^2 e}{\partial b_{ij} \partial z} - \frac{\partial^2 e}{\partial b_{ij} \partial \varepsilon^D_{kl}} {}^{D2}C^{ez}_{mn} + \frac{\partial^2 e}{\partial b_{ij} \partial \varepsilon^P_{kl}} {}^{P2}C^{eb}_{mn} \\ \frac{\partial^2 e}{\partial z \partial b_{ij}} - \frac{\partial^2 e}{\partial z \partial \varepsilon^D_{ij}} {}^{D1}C^{eb}_{mnkl} + \frac{\partial^2 e}{\partial z \partial \varepsilon^P_{ij}} {}^{P1}C^{eb}_{mnkl} & \frac{\partial^2 e}{\partial z \partial z} - \frac{\partial^2 e}{\partial z \partial \varepsilon^D_{ij}} {}^{D2}C^{ez}_{mn} + \frac{\partial^2 e}{\partial z \partial \varepsilon^P_{ij}} {}^{P2}C^{eb}_{mn} \\ \frac{\partial^2 e}{\partial \varepsilon^D_{ij} \partial b_{kl}} - \frac{\partial^2 e}{\partial \varepsilon^D_{ij} \partial \varepsilon^D_{kl}} {}^{D1}C^{eb}_{mnkl} + \frac{\partial^2 e}{\partial z \partial \varepsilon^P_{ij}} {}^{P1}C^{eb}_{mnkl} & \frac{\partial^2 e}{\partial \varepsilon^D_{ij} \partial z} - \frac{\partial^2 e}{\partial \varepsilon^D_{ij} \partial \varepsilon^D_{kl}} {}^{D2}C^{ez}_{mn} + \frac{\partial^2 e}{\partial z \partial \varepsilon^P_{ij}} {}^{P2}C^{eb}_{mn} \\ -{}^{D1}C^{eb}_{mnkl} & -{}^{D2}C^{ez}_{mn} \\ \frac{\partial^2 e}{\partial \varepsilon^P_{ij} \partial b_{kl}} - \frac{\partial^2 e}{\partial \varepsilon^P_{ij} \partial \varepsilon^P_{kl}} {}^{P1}C^{eb}_{mnkl} & \frac{\partial^2 e}{\partial \varepsilon^P_{ij} \partial z} - \frac{\partial^2 e}{\partial \varepsilon^P_{ij} \partial \varepsilon^P_{kl}} {}^{P2}C^{ez}_{mn} \\ {}^{P1}C^{eb}_{mnkl} & {}^{P2}C^{ez}_{mn} \\ {}^{D0}C^{eb}_{mnkl} & {}^{D0}C^{ez}_{mn} \\ {}^{P0}C^{eb}_{mnkl} & {}^{P0}C^{ez}_{mn} \end{Bmatrix} \begin{Bmatrix} \dot{b}_{ij} \\ \dot{z} \end{Bmatrix}$$

where

${}^{D1}C^{eb}_{mnkl} = \frac{\partial F^D}{\partial \chi^D} \frac{A^{D,eb}_{ij}}{B^{D,e}}$; ${}^{P1}C^{eb}_{mnkl} = \frac{\partial F^P}{\partial \chi^P} \frac{A^{P,eb}_{ij}}{B^{P,e}}$; ${}^{D2}C^{ez}_{mn} = \frac{\partial F^D}{\partial \chi^D} \frac{A^{D,ez}}{B^{D,e}}$; ${}^{P2}C^{ez}_{mn} = \frac{\partial F^P}{\partial \chi^P_{ij}} \frac{A^{P,ez}}{B^{P,e}}$; ${}^{D0}C^{eb}_{mnkl} = \frac{A^{D,eb}_{ij}}{B^{D,e}}$;
${}^{D0}C^{ez}_{mn} = \frac{A^{D,z}}{B^{D,e}}$; ${}^{P0}C^{eb}_{mnkl} = \frac{A^{P,eb}_{ij}}{B^{P,e}}$; ${}^{P0}C^{eZ}_{mn} = \frac{A^{P,ez}}{B^{P,e}}$.

Where substitutions for e , a_{ij} , b_{ij} , x , z are to be taken out of the appropriate column of the table 3. Specifically, the first two rows of the matrix in Eq.(35) explain the incremental relationships between the stresses, strains, temperature and entropy. The third and fourth rows are the evolution equations for the generalized damage stress and internal variable. The fifth and sixth rows are the evolution equations for the generalized plastic stress and internal variable. The final two rows allow evolution of the plastic multiplier and damage multyplier. The above solution only functions when plastic deformation occurs, i.e. $\dot{\varepsilon}^P_{ij} \neq 0$, $\dot{\varepsilon}^D_{ij} \neq 0$, and $\lambda^P > 0$, $\lambda^P > 0$. If the above solution results in $\lambda^P \leq 0$ and $\lambda^P \leq 0$, then it implies elastic unloading has occurred.

The elected formulation to use is determined by the application to a certain extent with personal preferences. The u and h formulations are particularly convenient for problems where changes in entropy are determined (e.g. adiabatic problems), while the f and g formulations are appropriate for those with prescribed temperature (e.g. isothermal problems). The u and f formulations corresponding to strain-space based on plasticity models, are particularly applicable when the strains are specified.

Conversely the h and g formulations corresponding with the more commonly used stress-spaced plasticity approaches, and are particularly convenient for problems with prescribed stresses.
However, by appropriate numerical manipulation it is possible to use any of the formulations for any application. For instance the g formulation leads directly to the compliance matrix. This can be naturally inverted to give the stiffness matrix.

$$[36]\ \left\{\begin{matrix} \dot{a}_{ij} \\ \dot{x} \\ -\dot{\chi}_{ij}^{D} \\ \dot{\varepsilon}_{ij}^{D} \\ -\dot{\chi}_{ij}^{P} \\ \dot{\varepsilon}_{ij}^{P} \\ \lambda^{D} \\ \lambda^{D} \end{matrix}\right\} = \left\{\begin{matrix} \frac{\partial^2 e}{\partial b_{ij}\partial b_{ij}} & \frac{\partial^2 e}{\partial b_{ij}\partial z} \\ \frac{\partial^2 e}{\partial z\partial b_{ij}} & \frac{\partial^2 e}{\partial z\partial z} \\ \frac{\partial^2 e}{\partial \varepsilon_{ij}^{D}\partial b_{ij}} & \frac{\partial^2 e}{\partial \varepsilon_{ij}^{D}\partial z} \\ 0 & 0 \\ \frac{\partial^2 e}{\partial \varepsilon_{ij}^{P}\partial b_{ij}} & \frac{\partial^2 e}{\partial \varepsilon_{ij}^{P}\partial z} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix}\right\} \left\{\begin{matrix} \dot{b}_{ij} \\ \dot{z} \end{matrix}\right\}.$$

5 CONCLUSION

the unified 32 thermodynamic formulation, which have Compatible with the basic principles of thermodynamics,is developed in this paper.The developed framework is capable to bring about guidance to build the model in according with the Law of Thermodynamics and concretely take more structural factors into account. The success of this approach depends on the comprehensive understanding of the micro-structural mechanisms of energy and dissipative relations of the system.
The main goal of a consistent and rigorous framework to the constitutive model of concrete has only been partly achieved in this study. However, the weakness of the proposed approach has also been pointed out, lying in the adoption of scale variables $\alpha_{ij}{}^{d}$, which can not reflect the damage-induced anisotropy. The incorporation of anisotropy features into the thermodynamic approach is of priority in the future research. This is required to capture the directional-induced responses of the material faithfully after the appearance of micro-cracks. In particular, both damage and plasticity parts of the model should account for the anisotropy in the post-peak behaviour. The introduction of damage variable as a tensor will require several modifications of the thermodynamic framework used in this study. Furthermore, the development of the rate-dependent effect evaluation, the combination of the simplified method and phenomenological plasticity, the choice of more facilitated potential and internal invariants forms were also expected and exploited.

REFERENCE

Bazant Z.P. 1978. On endochronic inelasticity and incremental plasticity. Int. J. Solids Struct , 14 :691–714.
Chaboche, J.L. 1997. Thermodynamic formulation of constitutive equations and application to the viscoelasticity and viscoplasticity of metals and polymers. Int. J. Solids Struct, 34 : 2239–2254.
Grassl P., Lundgren K. and Gylltoft K. 2002. Concrete in compression: a plasticity theory with a novel hardening law, Int. J. Solids Struct, 39: 5205–5223.
Houlsby, G.T.and Puzrin, A.M. 2000. A thermomechanical framework for constitutive models for rate-independent dissipative materials. Int. J. Plast. 16:1017–1047.
Krajcinovic D. 1985. Continuous damage mechanics revisited: basic concepts and definitions. J. Appl Mech , 52 :829–834.
L. Contrafatto, M. Cuomo, 2002. A new thermodynamically consistent continuum model for hardening plasticity coupled with damage. In.l J. of Solids and Structures ,:39 6241–6271.
Mazars, J. and Pijaudier-Cabot, G. 1989. Continuum damage theory-Application to concrete., Journal of Engineering Mechanics, ASCE,115 : 345-365.

Onate E., Oliver S. and Lubliner J. 1988. A constitutive model of concrete based on the incremental theory of plasticity. Engng Comput , 5 :309–319.
Simo, J.C. and Ju, J.W. 1987. Strain- and stress-based continuum damage models – I. Formulation. Int. J. Solids Struct, 23 :821-840.
Voyiadjis GZ. and Abu-Lebdeh TM. 1994. Plasticity model for concrete using the bounding surface concept. Int. J. Plasticity, 10 :1–21.
Ziegler, H. 1983. An Introduction to Thermomechanics, Amsterdam: North Holland